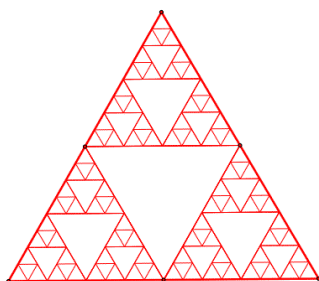


Higher Level Math Internal Assessment

Convergence of Remaining Area of Trisected Equilateral Triangle

Convergence of Remaining Area of Trisected Equilateral Triangle

The premise of this internal assessment arose from my initial interest in repeating shapes and objects which repeated themselves in a fractal-like manner, such as Sierpinski's Triangle and Carpet. Sierpinski's Triangle is a fractal pioneered by Waclaw Sierpinski in 1915, showcasing a self-similar structure that reiterates itself at multiple different levels, or magnifications (Parsons). To create the fractal, an equilateral triangle's sides are bisected and the midpoints are connected, thereby creating smaller equilateral triangles. This process is then repeated for the smaller equilateral triangles, and essentially has infinitely many possible iterations (Parsons).

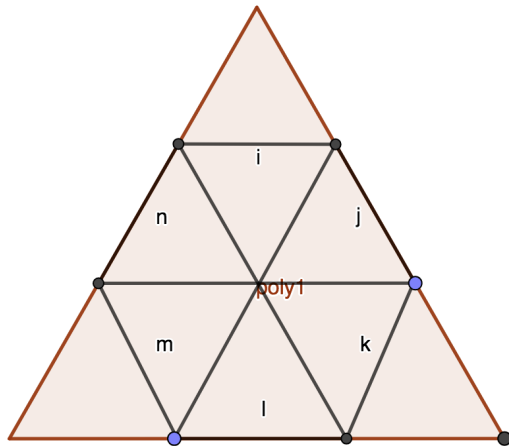
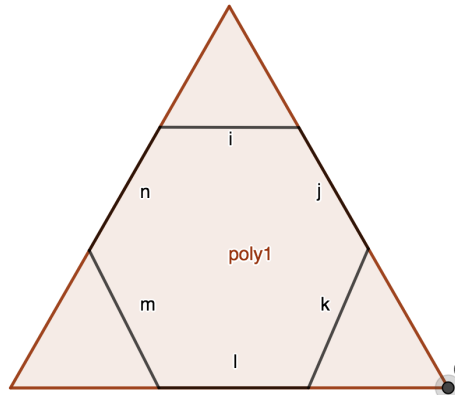


Sierpinski's Triangle

After doing further research about the discovery and principles of this fractal and the square version, Sierpinski's Carpet (Illustrative Mathematics), I wondered to myself what would happen if instead of bisecting the triangle's sides, we trisected it. As I experimented the diagram through various drawings, I realized that the corners of the triangles became increasingly obtuse and small, signifying that the interior area of the triangle left untouched would converge to some area. Thus, I sought to solve for the area if the process of subtracting the area of the trisected triangles was repeated infinitely many times.

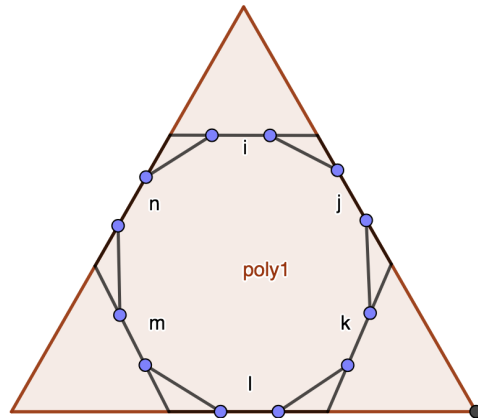
Let there be an equilateral triangle P_0 with area A . Each side of P_0 is trisected, and the corners are snipped off, creating a new polygon (in fact, a hexagon) P_1 . The process is continued again with P_1 – i.e. trisect each side and snip off the corners – to obtain a new polygon P_2 . Now repeat this process infinitely many times to create an object P_∞ . What is the area of P_∞ ?

First, let us consider when the triangle is trisected once, the total area being snipped off is equal to $\frac{1}{3}$ the area of the original triangle (P_1). Each of the corner equilateral triangles has side length $\frac{1}{3}$ of the original side length. The interior polygon is evidently a regular hexagon.

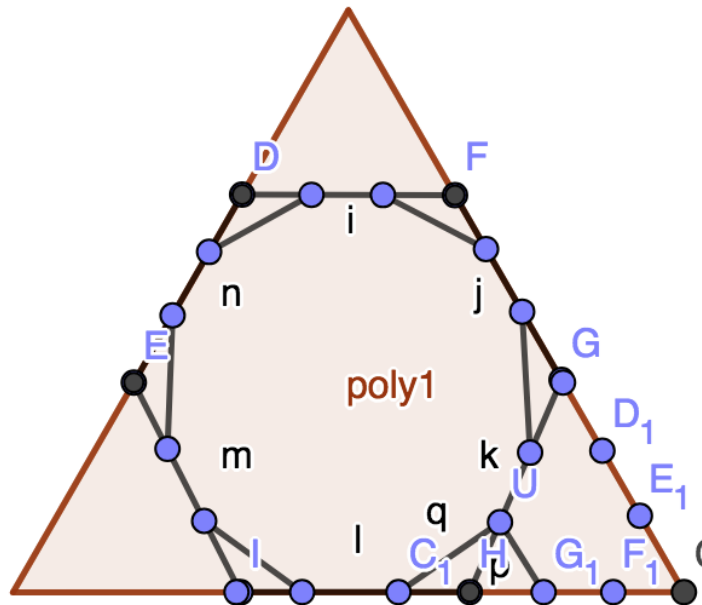


Since it is a regular hexagon, we can break it into six triangles. It is easy to see that the three triangles we took off when trisecting the triangle are $\frac{1}{3}$ of the total initial area. $\frac{2}{3}$ of the area is left.

Now, I can conjecture from this first iteration that each removed or trisected corner triangle is both $\frac{1}{9}$ the area of the previous iteration, and the number of triangles created by connecting the trisections is double the number from before. I will attempt to prove that the number of triangles being removed from P_{n+1} is double the number of triangles being cut from P_n and also that the area of each triangle removed from P_{n+1} Has $\frac{1}{9}$ the area of every triangle removed from P_n



Here, it is much more difficult to tell how much of the area is being cut off.



Focus on the bottom right corner of the triangle. We see that the area of C_1Hp equals the area of HG_1p as they have the same base length ($s/9$) and height. Now, we know $[OGp] = \frac{1}{2}$

$*Gp*po*\sin(GpO) = \frac{1}{2}(s/3)^2 * \sin(GpO)$ and we know that $[C_1Hp] = [HG_1p] =$

$\frac{1}{2} * (pG_1)(pH)\sin(GpO) = \frac{1}{2}(s/9)^2 * \sin(GpO)$. Thus $[C_1Hp] = \frac{1}{9}[OGp]$. So the area of a triangle being removed from $P_{n+1} = \frac{1}{9}$ the area of a triangle being removed from P_n . We can also see that the number of triangles being removed doubles from P_n to P_{n+1} . Thus the area

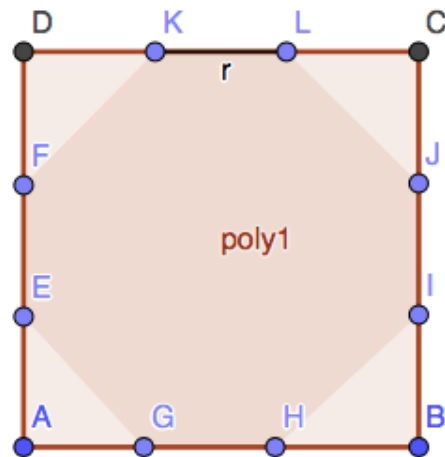
removed from $P_{n+1} = 2/9 \times$ the area removed from P_n . This exemplifies itself as an infinite geometric series with a common ratio of $2/9$. The sum of an infinite geometric series is $a/(1-r)$ where a is the initial value and r is the common ratio. If we write this out we see the total area removed from $P_\infty = (1/3) \times A + (1/3) \times (2/9) \times A + (1/3) \times (2/9)^2 \times A + \dots$

This is just a geometric series so the total area removed from $P_\infty = \frac{(1/3) \times A}{1 - 2/9} =$

$$\frac{(1/3) \times A}{7/9} = (3/7) \times A. \text{ So } [P_\infty] = A - (3/7) \times A = (4/7) \times A$$

Thus, the area of the interior polygon after infinitely many trisections and removals converges to $4/7$ of the original equilateral triangle's area.

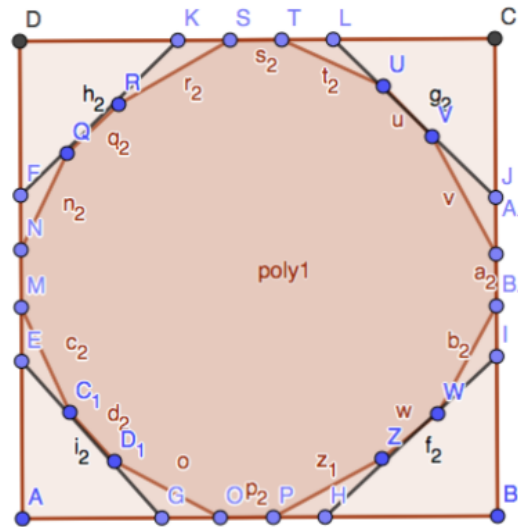
I was still curious and wanted to know how the convergence of the interior area would change in response to varying sides of a regular polygons. To try and see what would happen if I extend this process to any n -sided regular polygon, I took a square and trisected the sides as shown below:



The initial area we are cutting off of the square (P_1) is equal to

$$4 \times (1/2) \times (s/3)^2 \times \sin(90^\circ) = 2 \times (s/3)^2$$

Now when we trisect the sides of the octahedron P2 as shown in the figure below:



We can prove now as we did in part a that the area of each triangle we are cutting off of P_{n+1} is $1/9$ of the area of each triangle we cut off of P_n and the number of triangles being cut off from P_{n+1} is double the number of triangles that were cut from P_n . So the same geometric series as used with the equilateral triangle with ratio $2/9$ can be used to find the total area cut off the square just with a different starting area.

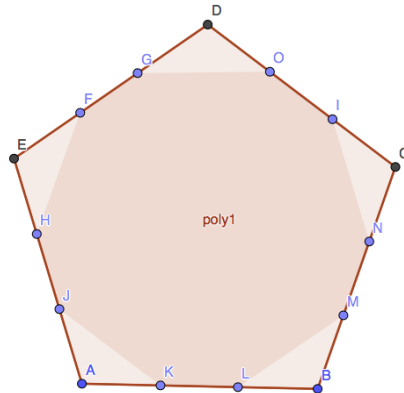
Thus the area of $P_\infty =$

A-2

$$* (s/3)^2 - (2/9) * 2 * (s/3)^2 - (2/9)^2 * 2 * (s/3)^2 \dots = A - \left(\frac{2 * (s/3)^2}{1 - (2/9)} \right) = A - \frac{2 * s^2}{7}$$

$$= s^2 - 2s^2/7 = 5s^2/7$$

We can now try this with a regular pentagon:

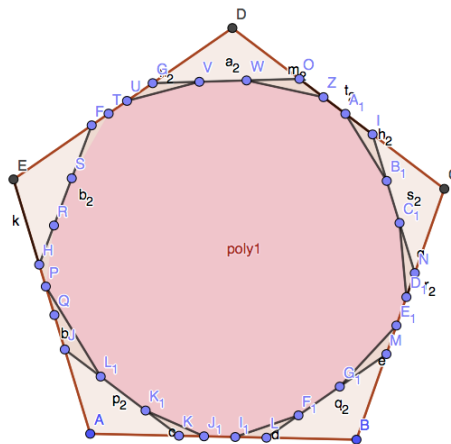


Each interior angle equals 108°

So the area of each triangle being removed from the polygon equals

$(1/2) * (s/3)^2 * \sin(108^\circ)$. And there are 5 triangles being removed so the total area being removed from the pentagon(P1) equals $5 * (1/2) * (s/3)^2 * \sin(108^\circ)$.

If we trisect each of the sides of the remaining decagon and connect the trisections and snip off the triangles as shown in the figure below:



We can prove now as we did in Part A that the area of each triangle we are cutting off of P_{n+1} is $1/9$ of the area of each triangle we cut off of figure P_n and the number of

triangles being cut off from P_{n+1} is double the number of triangles that were cut from P_n . So the same geometric series with ratio $2/9$ can be used to find the total area cut off just with a different starting area.

Thus the area of $P_\infty =$

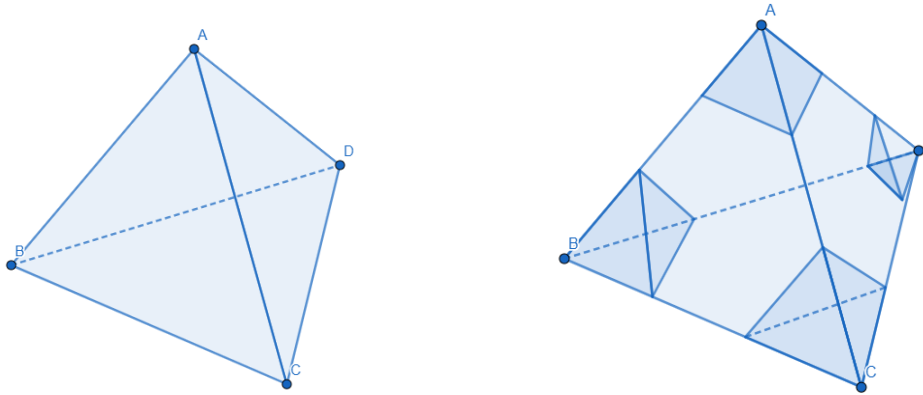
$A - x - x(2/9) - x(2/9)^2 - \dots = A - \frac{x}{1 - 2/9}$ where x = the area cut off from P_1 , which in this case equals $5 * (1/2) * (s/3)^2 * \sin(108^\circ)$.

We can now generalize this to any n -sided regular polygon with area A .

The area of $P_\infty = A - \frac{(n/2) * (s/3)^2 * \sin(\text{interior angle})}{1 - \frac{2}{9}}$ where the interior

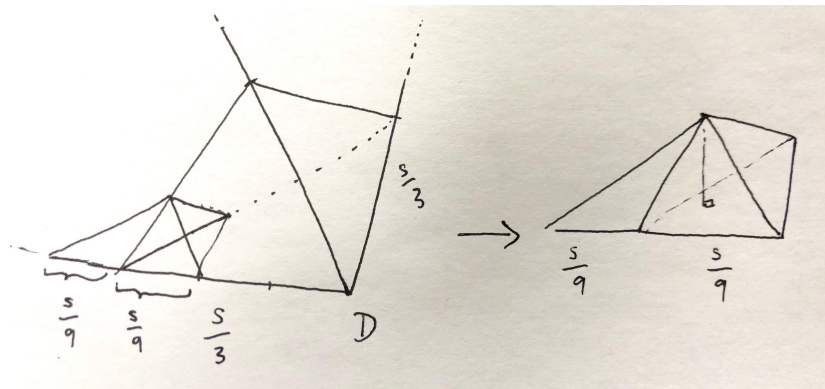
angle equals $\frac{180 * (n - 2)}{n}$

- a) Now repeat this process of trisection, but in three dimensions. It can be visualized with a pyramid with equilateral triangles as its faces.



As shown, similar triangular pyramids can be created by trisecting the edges of the original pyramid. Assuming that the side length of the edges are s , then the volume of the cut-off, smaller pyramid is $\frac{1}{3} * B * h = \frac{1}{3} * \sqrt{3}/4 * (\frac{s}{3})^2 * \frac{s}{3} * \sqrt{6}/3 = (\frac{s}{3})^3 * \frac{\sqrt{2}}{12}$, since the area of the equilateral triangle base is $\sqrt{3}/4 * (\frac{s}{3})^2$ and the height of a regular tetrahedron with side length $\frac{s}{3}$ is $(\frac{s}{3}) * \frac{\sqrt{6}}{3}$. Since there are 4 vertices, the total volume being cut off the first time is $4 * [(\frac{s}{3})^3 * \frac{\sqrt{2}}{12}]$

As this process is repeated, we can take a closer inspection at vertex D, where the next side is again being trisected.



Since the two pyramids have the same width, height, and depth, they therefore have the same volume. The volume of the second pyramid is $\frac{1}{3} * B * h = (\frac{s}{9})^3 * \frac{\sqrt{2}}{12}$, which is less than the previously cut off pyramid by a factor of $\frac{1}{3^3}$.

Since there are 3 vertices of the cutoff pyramids from which new ones can be created, the number of pyramids being cut off triples each time, so there are now $4*3 = 12$ pyramids, whereas the volume decreases by $1/27$. Therefore, the ratio of overall volume between successive cut-offs is $3*1/27 = 1/9$.

Again, the process of using a geometric series can be seen, as the original volume being cut off is $4 * [(\frac{s}{3})^3 * \frac{\sqrt{2}}{12}]$ and the ratio is $1/9$, so the total volume cut off is $a/(1-r) =$

$$\frac{4 * [(\frac{s}{3})^3 * \frac{\sqrt{2}}{12}]}{1 - 1/9} = \frac{5s^3\sqrt{2}}{72}$$

Now that the volume that is being subtracted is found, we can then solve for the volume remaining. Since the triangular pyramid has equilateral triangles as its faces, it is a regular tetrahedron. Thus its volume is $\frac{s^3}{6\sqrt{2}}$ (Varsity Tutors), and the convergence of the remaining volume is

$$\frac{s^3}{6\sqrt{2}} - \frac{5s^3\sqrt{2}}{72} = \frac{12\sqrt{2}s^3 - 5\sqrt{2}s^3}{72} = \frac{7s^3\sqrt{2}}{72}$$

This can be generalized to any regular polyhedra with N-vertices, since there are N number of pyramids with volume:

$$\frac{N * [(\frac{s}{3})^3 * \frac{\sqrt{2}}{12}]}{1 - 1/9}$$

Thus, I have solved for general expressions for 3-dimensional n-vertices polyhedron and 2-dimensional n-vertices regular polygons. Out of my own curiosity for the reiterating pattern of self-repeating patterns caused by trimming the exterior triangles made by trisections, I extended

solving for the convergence of the interior area of the original equilateral triangle to any integer number of sides for a regular polygon and any number of vertices for a three dimensional polyhedron.

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